MATHEMATICS

B.Sc, Part I (Paper II), Topic – problems on Pedal equation

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Question 1 Find the pedal equation of the parabola $y^2=4ax$ with regard to its focus.

Solution: The equation of the parabola is $y^2=4ax$ ——(1)

We know that the focus of the parabola is $(a,0)$.

Transfer the origin (0.0) to the focus $(a,0)$ of the parabola. Then the equation(1) of the parabola becomes

$$y^2=4a(x+a)$$

or $$y^2=4ax+4a^2 \qquad \text{—— —— (2)}$$

Making it homogeneous, we get

$$y^2=4axz+4a^2z^2, \text{ where } z=1.$$

or $$4a^2z^2+4axz-y^2=0$$

Let $$F(x,y,z)\equiv 4a^2z^2+4axz-y^2$$

$\therefore$ $$\frac{\partial F}{\partial x}=4az, \quad \frac{\partial F}{\partial y}=-2y,$$

$$\frac{\partial F}{\partial z}=8a^2z+4ax.$$

Therefore, the equation of the tangent to the parabola is

$$X\frac{\partial F}{\partial x}+Y\frac{\partial F}{\partial y}+Z\frac{\partial F}{\partial z}=0$$

or $$X.4az+Y.(-2y)+Z(8a^2z+4ax)=0$$

or $$2aX-Yy+4a^2+2ax=0$$

Now $p=$ the length of the perpendicular drawn from the origin $(0,0)$ upon this tangent.

$$=\frac{4a^2+2ax}{\sqrt{4a^2+y^2}}$$

or $\quad p = \dfrac{2a(2a+x)}{\sqrt{4a^2+4ax+4a^2}}$, from (2)

or $\quad p = \dfrac{2a(2a+x)}{\sqrt{4a(2a+x)}} = \sqrt{a(2a+x)}$.

or $\quad p^2 = a(2a+x)$. ——— (3)

Also $\quad r^2 = x^2+y^2$

$= x^2+4ax+4a^2 = (x+2a)^2$

or $\quad r = 2a+x$; or $r = \dfrac{p^2}{a}$ or $p^2 = ar$.

This is the required pedal equation.

Question 2. Prove that the pedal equation of the curve $x = ae^\theta(\sin\theta - \cos\theta)$, $y = ae^\theta(\sin\theta+\cos\theta)$ is $r = \sqrt{2}p$

Solution: Here

$$x = ae^\theta(\sin\theta - \cos\theta) \quad \text{——— (1)}$$
$$y = ae^\theta(\sin\theta + \cos\theta) \quad \text{——— (2)}$$

Differentiating (1) with respect to θ, we get

$$\frac{dx}{d\theta} = a[e^\theta(\sin\theta - \cos\theta) + e^\theta(\cos\theta + \sin\theta)]$$

$$= 2ae^\theta \sin\theta.$$

Differentiating (2) with respect to θ, we get

$$\frac{dy}{d\theta} = a[e^\theta(\sin\theta + \cos\theta) + e^\theta(\cos\theta - \sin\theta)]$$

$$= 2ae^\theta\cos\theta \,.$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^\theta\cos\theta}{2ae^\theta\sin\theta} = \cot\theta$$

The equation of the tangent is

$$Y - y = \frac{dy}{dx}(X - x).$$

or $Y - ae^{\theta}(\sin\theta + \cos\theta) = \cot\theta\,[X - ae^{\theta}(\sin\theta - \cos\theta)]$

or $Y\sin\theta - ae^{\theta}(\sin\theta + \cos\theta)\sin\theta$

$$= X\cos\theta - ae^{\theta}(\sin\theta - \cos\theta)\cos\theta$$

or $X\cos\theta - Y\sin\theta + ae^{\theta}(\sin^2\theta + \cos^2\theta) = 0$

or $X\cos\theta - Y\sin\theta + ae^{\theta} = 0$

$\therefore$ p = length of perpendicular from origin $(0,0)$ upon this tangent

$$= \frac{ae^{\theta}}{\sqrt{\cos^2\theta + \sin^2\theta}} = ae^{\theta}.$$

Also $r^2 = x^2 + y^2$

$$= a^2e^{2\theta}\left[(\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2\right] = 2a^2e^{2\theta}$$

or $r = \sqrt{2}ae^{\theta}$, or $r = \sqrt{2}p$.

This is the required pedal equation.

Question 3: Find the Pedal equation of the curve $r^n = a^n \sin n\theta$

Solution: We have $r^n = a^n \sin n\theta$. — —(1)

Diff. w.r. to θ, $nr^{n-1}\frac{dr}{d\theta} = na^n \cos n\theta$

i.e $$\frac{d\theta}{dr} = \frac{r^{n-1}}{a^n \cos n\theta} \quad \text{— — (2)}$$

We know that $\tan\phi = r\frac{d\theta}{dr} = r.\frac{r^{n-1}}{a^n\cos n\theta}$, from (2)

$$= \frac{r^n}{a^n \cos n\theta} = \frac{a^n \sin n\theta}{a^n \cos n\theta}, \text{ from (1)}$$

$$= \tan n\theta$$

$$\therefore \phi = n\theta \quad \text{———— (3)}$$

Also we have $p = r\sin\phi = r\sin n\theta$ from (3)

$$= r \cdot \frac{r^n}{a^n}, \quad \text{from (1)}$$

i.e., $pa^n = r^{n+1}$, which is required pedal equation.

(4) Question Obtain the pedal equation of the curve $r\sin\theta = a$.

Solution Differentiating $r\sin n\theta = a$ w.r.t. θ,

$$\frac{dr}{d\theta} \cdot \sin n\theta + nr\cos n\theta = 0$$

or $$\frac{1}{r} \cdot \frac{dr}{d\theta} = -n\frac{\cos n\theta}{\sin n\theta} = -n\cot n\theta$$

$$\therefore \cot\phi = -n\cot n\theta.$$

Now, $p = r\sin\phi = \dfrac{r}{\operatorname{cosec}\phi}$

$$= \frac{r}{\sqrt{1+\cot^2\phi}} = \frac{r}{\sqrt{1+n^2\cot^2 n\theta}} = \frac{r}{\sqrt{1+n^2 \cdot \frac{\cos^2 n\theta}{\sin^2 n\theta}}}$$

$$= \frac{r\sin n\theta}{\sqrt{\sin^2 n\theta + n^2(1-\sin^2\theta)}} = \frac{a}{\sqrt{\frac{a^2}{r^2} + n^2\left(1-\frac{a^2}{r^2}\right)}},$$

$$\{\because r\sin n\theta = a\}$$

$$\therefore p = \frac{ar}{\sqrt{a^2+n^2(r^2-a^2)}}$$